Complexiton Solutions of the Toda Lattice Equation

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Abstract

A set of coupled conditions consisting of differential-difference equations is presented for Casorati determinants to solve the Toda lattice equation. One class of the resulting conditions leads to an approach for constructing complexiton solutions to the Toda lattice equation through the Casoratian formulation. An analysis is made for solving the resulting system of differential-difference equations, thereby providing the general solution yielding eigenfunctions required for forming complexitons. Moreover, a feasible way is presented to compute the required eigenfunctions, along with examples of real complexitons of lower order.

Key words: Integrable lattice equation, Casorati determinant, spectral problem, soliton solution, complexiton solution
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1 Introduction

Many integrable equations, both continuous and discrete, have soliton solutions, which are exponentially decaying at spatial infinity. The existence of a three-soliton solution often indicates the integrability of the equation under investigation. The Toda lattice equation is one of the well-known lattice model equations exhibiting the soliton phenomenon [1]. Its multi-soliton solutions can be expressed through Casorati determinants [2, 3, 4], and its approximate soliton solutions have been explored around an exact soliton solution [5].

There are also positons and negatons to the Toda lattice equation, which can be presented by generalized Casorati determinants [6, 7], and solitons are just a specific class of negatons. Positons and negatons were first presented for the Korteweg-de Vries (KdV) equation (see, for example, [8, 9]). All three classes of solutions–solitons, positons and negatons–are associated with real eigenvalues of the associated spectral problems. Moreover, the absolute values of soliton and negaton solutions contain one kind of elementary transcendental functions - exponential functions of the space variables, and the absolute values of positon solutions contain another kind of elementary transcendental functions - trigonometrical functions of the space variables.

A challenging problem in solution theory is how to construct a different kind of explicit exact solutions to soliton equations, whose absolute values involve both exponential and trigonometrical functions of the space variables and which are associated with the complex eigenvalues of the associated spectral problems. The absolute values do not need to be taken if solutions are real, as in the case of the KdV equation; but do need to be taken if solutions are complex, as in the case of the nonlinear Schrödinger equation. Exact solutions of such kind are called complexiton solutions and have been presented for the KdV equation [10]. Note that interaction solutions between positons and negatons can also contain both exponential and trigonometrical functions of the space variables, but they are associated with real eigenvalues of the associated spectral problems and thus they are not examples of so-called complexitons. Although these solutions belong to a broader class of exact solutions, they can be well formulated once three kinds of basic solutions–negatons, positons and complexitons–are presented [11].

Therefore, for the Toda lattice equation, the basic question for us is whether there exist complexiton solutions and how one can construct complexitons if they exist. This is the topic that we would like to address in this paper. It is known that the Casoratian formulation is a powerful technique to generate explicit solutions of integrable lattice

equations [3, 4]. Solutions determined by the Casorati determinant technique and generalized Casorati determinant technique are called Casorati determinant solutions and generalized Casorati determinant solutions, respectively [7]. For the Toda lattice equation, solitons are examples of Casorati determinant solutions [2, 3], and positons and nagatons are examples of generalized Casorati determinant solutions [7].

In this paper, we would like to show that there exist complexiton solutions of the Toda lattice equation through the Casoratian formulation. Inspired by its Lax pair, a set of coupled conditions will be presented for guaranteeing Casorati determinants to be solutions of the Toda lattice equation in Section 2, and this yields an approach to a broad class of Casorati determinant solutions and generalized Casorati determinant solutions of the Toda lattice equation. The resulting coupled conditions will be used to construct real complexiton solutions to the Toda lattice equation in Section 3. Moreover, a feasible way will be proposed to construct sets of special eigenfunctions satisfying the required conditions in Section 4, together with concrete examples of real complexitons of lower order. A few concluding remarks will be given in Section 5.

2 Casoratian formulation

Let us consider the Toda lattice equation in the following form:

$$\dot{a}_n = a_n(b_{n-1} - b_n)\,, \quad \dot{b}_n = a_n - a_{n+1}\,, \tag{2.1}$$

where (also in the rest of the paper) the dot denotes the differentiation with respect to the time variable t. This Toda lattice equation can be reduced to the periodic case ($a_{n+N} = a_n$ and $b_{n+N} = b_n$ for some positive integer N) and the finite case (only finitely many a_n and b_n are non-zero). It is also more general than the square form of the Toda lattice equation [12]:

$$\dot{a}_n = a_n(b_n - b_{n+1})\,, \quad \dot{b}_n = 2(a_{n-1}^2 - a_n^2)\,, \tag{2.2}$$

because there is a solution transformation

$$(a_n(t), b_n(t)) \to \big((a_{n-1}(\frac{1}{2}t))^2, b_n(\frac{1}{2}t)\big)$$

from the square form (2.2) to the non-square form (2.1). On the other hand, the Toda lattice equation (2.1) is the isospectral ($\lambda_t = 0$) compatibility condition of the following

spectral problems:

$$\begin{cases} \dot{\phi}(n) = b_{n-1}\phi(n) + \phi(n-1), \\ a_n\phi(n+1) + b_{n-1}\phi(n) + \phi(n-1) = \lambda\phi(n), \end{cases} \tag{2.3}$$

where λ is a spectral parameter. Namely, it has the Lax representation:

$$\dot{L} = [A, L], \tag{2.4}$$

where the Lax pair is defined by

$$\begin{cases} L_{nm} = a_n\delta_{n+1,m} + b_{n-1}\delta_{nm} + \delta_{n-1,m}, \\ A_{nm} = \delta_{n+1,m} + b_{n-1}\delta_{nm}. \end{cases} \tag{2.5}$$

Under an dependent variable transformation

$$a_n = 1 + \frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad b_n = \frac{d}{dt}\log\frac{\tau_n}{\tau_{n+1}} = \frac{\dot{\tau}_n\tau_{n+1} - \tau_n\dot{\tau}_{n+1}}{\tau_n\tau_{n+1}}, \tag{2.6}$$

we have

$$\begin{cases} \dot{a}_n - a_n(b_{n-1} - b_n) = 0, \\ \dot{b}_n - a_n + a_{n+1} = \dfrac{\ddot{\tau}_n\tau_n - (\dot{\tau}_n)^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2}{\tau_n^2} - \dfrac{\ddot{\tau}_{n+1}\tau_{n+1} - (\dot{\tau}_{n+1})^2 - \tau_{n+2}\tau_n + \tau_{n+1}^2}{\tau_{n+1}^2}, \end{cases}$$

and thus the Toda lattice equation (2.1) can be satisfied if we require the bilinear equation

$$[\frac{1}{2}D_t^2 - 2\sinh^2(\frac{D_n}{2})]\tau_n \cdot \tau_n = \ddot{\tau}_n\tau_n - (\dot{\tau}_n)^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 = 0, \tag{2.7}$$

where D_t and D_n are Hirota's operators. This is called the bilinear Toda lattice equation. Through the dependent variable transformation (2.6), multi-soliton solutions of the Toda lattice equation (2.1) can be presented by the Casorati determinant [3, 13]:

$$\mathrm{Cas}(\phi_1(n), \phi_2(n), \cdots, \phi_N(n)) := \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}, \ N \geq 1, \tag{2.8}$$

provided that the functions $\phi_i(n)$, $1 \leq i \leq N$, solve

$$\phi_i(n+1) + \phi_i(n-1) = \lambda_i\phi_i(n), \quad (\phi_i(n))_t = \phi_i(n-1), \quad 1 \leq i \leq N \tag{2.9}$$

where $\lambda_i = 2\cosh(k_i)$ and the k_i's are arbitrary distinct real constants. The conditions in (2.9) correspond to the case of the spectral problems (2.3) with $a_n = 1$ and $b_n = 0$, a special solution to the Toda lattice equation (2.1). This also implies that the Casorati determinant solution is actually resulted from the Darboux transformation of the Toda lattice equation.

In what follows, we would like to show that the Casorati determinant presents a very broad class of exact solutions to the Toda lattice equation (2.1), among which solitons, positons and negatons are special examples. The following theorem is a generalization to the cases of solitons, positons and negatons.

Theorem 2.1. *Assume that a set of functions $\phi_i(n) = \phi_i(n,t), 1 \le i \le N$, solve a system of differential-difference equations*

$$\phi_i(n+1) + \phi_i(n-1) = \sum_{j=1}^{N} \lambda_{ij}\phi_j(n)\,, \quad 1 \le i \le N\,, \tag{2.10}$$

$$\partial_t \phi_i(n) = \phi_i(n+\delta)\,, \quad 1 \le i \le N\,, \tag{2.11}$$

where $\delta = \pm 1$ and the λ_{ij}'s are arbitrary constants. Then the Casorati determinant

$$\tau_n = \mathrm{Cas}(\phi_1(n), \cdots, \phi_N(n))$$

gives a solution to the bilinear Toda lattice equation (2.7), and further the dependent variable transformation (2.6) presents a solution to the Tada lattice equation (2.1).

Proof. We only prove the result under (2.11) with $\delta = -1$. The other case with $\delta = 1$ is completely similar. Assuming that

$$\Phi_N(n) = (\phi_1(n), \cdots, \phi_N(n))^T, \tag{2.12}$$

we adopt the notation

$$|i_1, i_2, \cdots, i_N| := \det(\Phi_N(n+i_1), \Phi_N(n+i_2), \cdots, \Phi_N(n+i_N))\,,\ k..l := k, k+1, \cdots, l\,, \tag{2.13}$$

where $i_1, i_2, \cdots, i_N$ and $k < l$ are arbitrary integers. For example, we have

$$|0..N-2, N| = \det(\Phi_N(n), \Phi_N(n+1), \cdots, \Phi_N(n+N-2), \Phi_N(n+N)),$$
$$|-1, 1..N-1| = \det(\Phi_N(n-1), \Phi_N(n+1), \Phi_N(n+2), \cdots, \Phi_N(n+N-1)).$$

Directly from the conditions in (2.11) with $\delta = -1$, we obtain the expressions for the first two derivatives of the τ-function τ_n with respect to t:

$$\dot{\tau}_n = |-1, 1..N-1|\,,\ \ddot{\tau}_n = |-2, 1..N-1| + |-1, 0, 2..N-1|\,. \tag{2.14}$$

On the other hand, we have the general result for any determinant $|A_{ij}|$:

$$\sum_{k=1}^{N} |A_{ij}|_k = \sum_{k=1}^{N} |A_{ij}|^k \,, \tag{2.15}$$

where $|A_{ij}|_k$ denotes the determinant A_{ij} with its k-th row operated by the operator S:

$$(S\phi)(n) := \phi(n+1) + \phi(n-1), \tag{2.16}$$

and $|A_{ij}|^k$ denotes the determinant with its k-th column operated by the operator S. Applying (2.15) to two determinants $|0..N-1|$ and $|-1,1..N-1|$ and using the conditions in (2.10), we obtain the determinant identities:

$$\sum_{i=1}^{N} \lambda_{ii}|0..N-1| = |0..N-2,N| + |-1,1..N-1| \,, \tag{2.17}$$

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_{ii}|-1,1..N-1| = |0..N-1| + |-2,1..N-1| \\ &+|-1,0,2..N-1| + |-1,1..N-2,N| \,. \end{aligned} \tag{2.18}$$

Now, making use of (2.14), (2.17) and (2.18), we find that the left-hand side of (2.7) gives the terms

$$\begin{aligned} &\ddot{\tau}_n\tau_n - (\dot{\tau}_n)^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 \\ &= \big(|-2,1..N-1| + |-1,0,2..N-1|\big)|0..N-1| - |-1,1..N-1|^2 \\ &\quad -|1..N||-1..N-2| + |0..N-1|^2 \\ &= \Big(\sum_{i=1}^{N} \lambda_{ii}|-1,1..N-1| - |0..N-1| - |-1,1..N-2,N|\Big)|0..N-1| \\ &\quad -\Big(\sum_{i=1}^{N} \lambda_{ii}|0..N-1| - |0..N-2,N|\Big)|-1,1..N-1| \\ &\quad -|1..N||-1..N-2| + |0..N-1|^2 \\ &= -|-1,1..N-2,N||0..N-1| + |0..N-2,N||-1,1..N-1| - |1..N||-1..N-2| \,. \end{aligned}$$

The last sum above is the Laplace expansion by $N\times N$ minors of the following $2N\times 2N$ determinant

$$-\frac{1}{2}\left|\begin{array}{c:c:cccc} 1..N-2 & \emptyset & -1 & 0 & N-1 & N \\ \hdashline \emptyset & 1..N-2 & -1 & 0 & N-1 & N \end{array}\right| ,$$

where $\emptyset$ indicates the $N\times(N-2)$ zero matrix. This can be easily shown to be identically zero. Thus, the solution is verified. □

Note that the first half conditions in (2.9) are just a special case of the conditions in (2.10). Therefore, we can expect to get more solutions to the Toda lattice equation (2.1) by solving the system of differential-difference equations, (2.10) and (2.11), as in the KdV case [11]. Moreover, the entire problem of constructing explicit solutions is reduced to the problem of solving the system, (2.10) and (2.11).

The system of (2.10) and (2.11) can be compactly written as

$$(S\Phi_N)(n) \equiv \Phi_N(n+1) + \Phi_N(n-1) = \Lambda\Phi_N(n), \ (\Phi_N(n))_t = \Phi_N(n+\delta), \tag{2.19}$$

where Φ_N is defined by (2.12) and

$$\Lambda := (\lambda_{ij})_{N\times N} \tag{2.20}$$

is called the coefficient matrix of the system of (2.10) [or the system of (2.10) and (2.11)]. Note that a constant similar transformation for the coefficient matrix Λ does not change the resulting Casorati determinant solution to the Toda lattice equation (2.1). Actually, if we have $M = P^{-1}\Lambda P$ for some constant invertible matrix P, then $\tilde{\Phi}_N = P\Phi_N$ satisfies

$$(S\tilde{\Phi}_N)(n) \equiv \tilde{\Phi}_N(n+1) + \tilde{\Phi}_N(n-1) = M\tilde{\Phi}_N(n), \ (\tilde{\Phi}_N(n))_t = \tilde{\Phi}_N(n+\delta). \tag{2.21}$$

Obviously, the dependent variable transformation (2.6) leads to the same Casorati determinant solutions from Φ_N and $\tilde{\Phi}_N$. Therefore, by linear algebra, we only need to consider the following two types of Jordan blocks of the coefficient matrix Λ:

$$\begin{bmatrix} \lambda_j & & & 0 \\ 1 & \lambda_j & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & \lambda_j \end{bmatrix}_{k_j\times k_j}, \tag{2.22}$$

$$\begin{bmatrix} A_j & & & 0 \\ I_2 & A_j & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & I_2 & A_j \end{bmatrix}_{l_j\times l_j}, \ A_j = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix}, \ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.23}$$

where λ_j, α_j and $\beta_j > 0$ are all real constants, and k_j and l_j are positive integers. The first type of Jordan blocks has the real eigenvalue λ_j with algebraic multiplicity k_j, and the second type of Jordan blocks has the complex eigenvalues $\lambda_{j,\pm} = \alpha_j \pm \beta_j i$ with algebraic multiplicity l_j.

The case of real eigenvalues corresponds to positons, nagatons and rational solutions [7, 14]. In what follows, we will focus on the case of complex eigenvalues to present complexitons. We will show how to solve the system of differential-difference equations, (2.10) and (2.11), in the case of complex eigenvalues such that the Casoratian formulation leads to real complexiton solutions of the Toda lattice equation.

3 Complexiton solutions

In order to construct complexitons, let us begin to solve the system of differential-difference equations, (2.10) and (2.11), whose coefficient matrix consists of Jordan blocks of the second type. Since all subsystems corresponding to Jordan blocks are separated, it suffices to illustrate how to solve a system associated with one Jordan block of the second type. Let us specify such a system as

$$(S\Phi)(n) \equiv \Phi(n+1) + \Phi(n-1) = \Lambda\Phi(n),\ (\Phi(n))_t = \Phi(n+\delta), \tag{3.1}$$

where $\delta = \pm 1$ and

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{2l-1} \\ \phi_{2l} \end{bmatrix},\ \Lambda = \begin{bmatrix} A & & & 0 \\ I_2 & A & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & I_2 & A \end{bmatrix}_{l\times l},\ A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{3.2}$$

If we introduce

$$\lambda = \alpha + \beta i,\ F_k = \phi_{2k-1} + \phi_{2k} i,\ 1 \le k \le l, \tag{3.3}$$

then the system (3.1) is obviously equivalent to the following triangular system for all F_k:

$$SF_k = \lambda F_k + F_{k-1},\ (F_k(n))_t = F_k(n+\delta),\ 1 \le k \le l, \tag{3.4}$$

where $F_0 = 0$.

Lemma 3.1. *Let λ be a complex number not equal 2 and $\delta = \pm 1$. Then the homogeneous system of differential-difference equations*

$$(Sf)(n) \equiv f(n+1) + f(n-1) = \lambda f(n),\ (f(n))_t = f(n+\delta), \tag{3.5}$$

has its general solution

$$f(\lambda;c,d)(n) \equiv f(n) = c\omega^n e^{t\omega^\delta} + d\omega^{-n} e^{t\omega^{-\delta}}, \tag{3.6}$$

where ω is defined by

$$\lambda = \omega + \omega^{-1},\ \ i.e.,\ \ \omega^2 - \lambda\omega + 1 = 0, \tag{3.7}$$

and c and d are arbitrary constants.

Proof. Note that the general solution to the linear difference equation

$$(Sf)(n) \equiv f(n+1) + f(n-1) = \lambda f(n) = (\omega + \omega^{-1}) f(n) \tag{3.8}$$

has two free parameters. Moreover, it is easy to show that $f(n) = \omega^n$ and $f(n) = \omega^{-n}$ are two solutions to (3.8), and they are linearly independent since $\lambda \neq 2$. Hence, the general solution to the difference equation (3.8) is given by

$$f(n) = c_1(t)\omega^n + d_1(t)\omega^{-n},$$

where c_1 and d_1 are two functions of t. On the other hand, the differential equation $(f(n))_t = f(n+\delta)$ requires that

$$\dot{c}_1 = c_1\omega^\delta,\ \ \dot{d}_1 = d_1\omega^{-\delta},$$

and thus we have

$$c_1 = c\mathrm{e}^{t\omega^\delta},\ \ d_1 = d\mathrm{e}^{t\omega^{-\delta}},$$

where c and d are arbitrary constants. Therefore, the general solution to the differential-difference equation (3.5) is given by (3.6). The proof is finished. □

Remark: The condition $\lambda = 2$ is equivalent to $\omega = 1$, and moreover, ω^n and ω^{-n} are linearly dependent if and only if $\omega = 1$. Therefore, $\lambda \neq 2$ is necessary for guaranteeing the linear independence of ω^n and ω^{-n}. Actually, the case of $\lambda = 2$ corresponds to rational solutions [14].

Theorem 3.1. *Let λ be a complex number not equal 2 and $\delta = \pm 1$. Suppose that $f(\lambda;c,d)$ is the general solution to (3.5), defined by (3.6), and for each $1 \leq k \leq l$, define $f_k = f(\lambda;c_k,d_k)$ with a pair of arbitrary constants c_k and d_k. Then the general solution to the triangular system of differential-difference equations (3.4) is given by*

$$F_k = \sum_{p=0}^{k-1} \frac{1}{p!} \frac{\partial^p f_{k-p}}{\partial \lambda^p} = \sum_{p=0}^{k-1} \frac{1}{p!} \frac{\partial^p f(\lambda;c_{k-p},d_{k-p})}{\partial \lambda^p},\ \ 1 \leq k \leq l. \tag{3.9}$$

Proof. First, from (3.5), we have

$$Sf_k = \lambda f_k,\ (f_k(n))_t = f_k(n+\delta),\ 1 \le k \le l.$$

Differentiating these equalities p times with respect to λ, we obtain

$$\begin{cases} S(\frac{1}{p!}\frac{\partial^p f_k}{\partial \lambda^p}) = \lambda(\frac{1}{p!}\frac{\partial^p f_k}{\partial \lambda^p}) + \frac{1}{(p-1)!}\frac{\partial^{p-1} f_k}{\partial \lambda^{p-1}},\ 1 \le k \le l,\ p \ge 1, \\ \big[(\frac{1}{p!}\frac{\partial^p f_k}{\partial \lambda^p})(n)\big]_t = (\frac{1}{p!}\frac{\partial^p f_k}{\partial \lambda^p})(n+\delta),\ 1 \le k \le l,\ p \ge 1. \end{cases} \tag{3.10}$$

Second, note that the linear system (3.4) is triangular, and so we can solve the system one by one from F_1 through F_l.

By Lemma 3.1, the general solution to the first subsystem for F_1 in (3.4) can be defined by f_1 with a pair of arbitrary constants c_1 and d_1. Now let $2 \le k \le l$ (if $l = 1$, we are finished). Assume that we already solve the first $k-1$ subsystems for F_p, $1 \le p \le k-1$. Then the k-th subsystem for F_k in (3.4) can be viewed as a non-homogeneous linear system and thus its general solution is determined by

$$F_k = F_{k,h} + F_{k,s},$$

where $F_{k,h}$ is the general solution to the homogeneous counterpart of the k-th subsystem and $F_{k,s}$ is a special solution to the non-homogeneous k-th subsystem. Again by Lemma 3.1, the general solution $F_{k,h}$ to the k-th subsystem of (3.4) can be defined by f_k with a pair of arbitrary constants c_k and d_k. On the other hand, by an inspection, a special solution $F_{k,s}$ to the k-th subsystem can be found to be

$$F_{k,s} = \sum_{p=1}^{k-1} \frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \lambda^p}.$$

This can be proved by using (3.10). Actually, we have

$$\begin{aligned} (S-\lambda)F_{k,s} &= \sum_{p=1}^{k-1}(S-\lambda)\big(\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \lambda^p}\big) \\ &= \sum_{p=1}^{k-1}\frac{1}{(p-1)!}\frac{\partial^{p-1} f_{k-p}}{\partial \lambda^{p-1}} = \sum_{p=0}^{k-2}\frac{1}{p!}\frac{\partial^p f_{k-p-1}}{\partial \lambda^p} = F_{k-1}, \\ \big(F_{k,s}(n)\big)_t &= \sum_{p=1}^{k-1}\big[(\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \lambda^p})(n)\big]_t \\ &= \sum_{p=1}^{k-1}\big(\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \lambda^p}\big)(n+\delta) = F_{k,s}(n+\delta). \end{aligned}$$

Therefore, the above function $F_{k,s}$ is a special solution to the k-th subsystem. Then, it follows that the general solution to the k-th subsystem of (3.4) is given by

$$F_k = F_{k,h} + F_{k,s} = f_k + \sum_{p=1}^{k-1} \frac{1}{p!} \frac{\partial^p f_{k-p}}{\partial \lambda^p} = \sum_{p=0}^{k-1} \frac{1}{p!} \frac{\partial^p f_{k-p}}{\partial \lambda^p}.$$

Finally, an induction ensures that the general solution to the system (3.4) is given by (3.9). The proof is finished. □

Theorem 3.1 provides us with an approach for solving a system of differential-difference equations, (2.10) and (2.11), whose coefficient matrix Λ consists of Jordan blocks of the second type. Once we solve the system of (2.10) and (2.11), it follows from Theorem 2.1 that the corresponding Casorati determinant gives us a solution to the Toda lattice equation. If the coefficient matrix Λ consists of m Jordan blocks of the second type in (2.23), then the Casorati determinant solution reads as

$$\begin{cases} a_n = 1 + \dfrac{d^2}{dt^2} \log \tau_n, \ \ b_n = \dfrac{d}{dt} \log \dfrac{\tau_n}{\tau_{n+1}}, \\ \tau_n = \mathrm{Cas}\big(\phi_1(n), \cdots, \phi_{2l_1}(n); \cdots; \phi_{2(l_1+\cdots+l_{m-1})+1}(n), \cdots, \phi_{2(l_1+\cdots+l_m)}(n)\big), \end{cases} \tag{3.11}$$

where the involved eigenfunctions are determined by the formula (3.9) with $\lambda_j = \alpha_j + \beta_j i$, $1 \le j \le m$.

In the following, we would like to show that the solutions defined by (3.11) are complexiton solutions. To this end, let us write

$$\lambda = 2\cosh \mu = \mathrm{e}^{\mu} + \mathrm{e}^{-\mu}, \ \ \mu \in \mathbb{C} \tag{3.12}$$

and thus we can have

$$\omega = \mathrm{e}^{\mu}, \ \ \mu \in \mathbb{C}. \tag{3.13}$$

Note that while μ goes over the complex field, $\lambda = 2\cosh\mu$ will exhaust all complex values, and thus the assumption (3.12) does not lose generality. Then by Lemma 3.1, the general solution of the system (3.5) is given by

$$f(n) = c\exp(\mu n + t\exp(\delta\mu)) + d\exp(-\mu n + t\exp(-\delta\mu)), \tag{3.14}$$

where c and d are arbitrary constants. The other selection of $\omega = \mathrm{e}^{-\mu}$ leads to the same solution of the system (3.5). Now write

$$f = \phi_1 + \phi_2 i, \ \ \mu = a + bi, \ \ a \in \mathbb{R}, \ \ b \neq 0 \in \mathbb{R}, \tag{3.15}$$

and assume that c and d are real constants but $c^2+d^2 \neq 0$ in order that $f \neq 0$. Then, we have

$$\lambda = \alpha + \beta i = 2(\cosh a \cos b) + 2(\sinh a \sin b) i. \tag{3.16}$$

Moreover, the system (3.5) becomes the following system

$$\begin{cases} \phi_1(n+1) + \phi_1(n-1) = 2(\cosh a \cos b)\phi_1(n) - 2(\sinh a \sin b)\phi_2(n), \\ \phi_2(n+1) + \phi_2(n-1) = 2(\sinh a \sin b)\phi_1(n) + 2(\cosh a \cos b)\phi_2(n), \\ (\phi_j(n))_t = \phi_j(n+\delta), \ \ j=1,2, \end{cases} \tag{3.17}$$

and its solution (ϕ_1, ϕ_2) reads as

$$\begin{cases} \phi_1(n) = (\phi_1(a,b;c,d))(n) := \mathrm{Re}(f(n)) \\ = c e^{na + te^{\delta a}\cos \delta b} \cos(nb + te^{\delta a}\sin \delta b) + d e^{-na+te^{-\delta a}\cos \delta b} \cos(nb + te^{-\delta a}\sin \delta b)\,, \\ \phi_2(n) = (\phi_2(a,b;c,d))(n) := \mathrm{Im}(f(n)) \\ = c e^{na + te^{\delta a}\cos \delta b} \sin(nb + te^{\delta a}\sin \delta b) - d e^{-na+te^{-\delta a}\cos \delta b} \sin(nb + te^{-\delta a}\sin \delta b)\,. \end{cases} \tag{3.18}$$

The initial set (2.10) of difference equations with the second type of Jordan blocks [i.e., Jordan blocks in (2.23)] tells us that the solutions defined by (3.11) are associated with the complex eigenvalues of the associated spectral problem. The expressions (3.18) and (3.9) for the required eigenfunctions indicate that the resulting real solutions contain both exponential and trigonometric functions of the space variable n. Therefore, it follows that the solutions determined by (3.11) are real complexitons to the Toda lattice equation (2.1), which establishes the following theorem.

Theorem 3.2. *Let α_j and β_j, $1 \leq j \leq m$, be real numbers and $\beta_j \neq 0$, $1 \leq j \leq m$. Then the Toda lattice equation (2.1) has a class of real complexiton solutions determined by the formula (3.11) and the formula (3.9) with $\lambda = \lambda_j = \alpha_j + \beta_j i$, $1 \leq j \leq m$.*

A solution defined by (3.11) is called an m-complexiton solutions (or simply, an m-complexiton) of order $(l_1-1, l_2-1, \cdots, l_m-1)$ to the Toda lattice equation (2.1). If $l_j = 1$, $1 \leq j \leq m$ or $m=1$, we simply say an m-complexiton solution or a single complextion solution of order l_1-1. Based on the expressions of eigenfunctions in (3.9), we see that the order $(l_1-1, l_2-1, \cdots, l_m-1)$ of the complexiton reflect the maximum orders of derivatives of eigenfunctions with respect to the corresponding eigenvalues.

4 Construction of examples

Theorems 2.1 and 3.1 provide a general solution procedure using a general set of eigenfunctions. It is, however, easier to apply some other techniques to present concrete complexitons.

The simplest way to get a special set of eigenfunctions required in complexitons is to take only one term in the expressions (3.9) for each F_k. This can be realized as follows. We set

$$\Phi_2 = (\phi_1, \phi_2)^T$$

as before, and assume that $f = \phi_1 + \phi_2 i$ solves the system (3.5) with $\lambda = \alpha + \beta i$. Then the choice of $c_k = d_k = 0$, $2 \le k \le l$, presents a special set of eigenfunctions

$$\big(\Phi_2^T(\lambda), \frac{1}{1!}\partial_\lambda \Phi_2^T(\lambda), \cdots, \frac{1}{(l-1)!}\partial_\lambda^{l-1}\Phi_2^T(\lambda)\big)^T \tag{4.1}$$

to the system of (2.10) and (2.11) with the coefficient matrix Λ:

$$\Lambda = \begin{bmatrix} A & & & 0 \\ I_2 & A & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & I_2 & A \end{bmatrix}_{l\times l}, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Thus, a class of generalized Casorati determinant solutions to the Toda lattice equation (2.1) can be generated from the pair of eigenfunctions $\Phi_2 = (\phi_1, \phi_2)^T$ as follows:

$$\begin{cases} a_n = 1 + \dfrac{d^2}{dt^2} \log \mathrm{Cas}\big(\Phi_2^T(n), \frac{1}{1!}\partial_\lambda \Phi_2^T(n), \cdots, \frac{1}{(l-1)!}\partial_\lambda^{l-1}\Phi_2^T(n)\big), \\ b_n = \dfrac{d}{dt} \log \dfrac{\mathrm{Cas}\big(\Phi_2^T(n), \frac{1}{1!}\partial_\lambda \Phi_2^T(n), \cdots, \frac{1}{(l-1)!}\partial_\lambda^{l-1}\Phi_2^T(n)\big)}{\mathrm{Cas}\big(\Phi_2^T(n+1), \frac{1}{1!}\partial_\lambda \Phi_2^T(n+1), \cdots, \frac{1}{(l-1)!}\partial_\lambda^{l-1}\Phi_2^T(n+1)\big)}. \end{cases}$$

A more general generalized Casorati determinant solution can be constructed by combining pairs of eigenfunctions $\Phi_2(\lambda_1), \Phi_2(\lambda_2), \cdots, \Phi_2(\lambda_m)$ associated with complex eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$, respectively. If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ have algebraic multiplicities $l_1, l_2, \cdots, l_m$, respectively, the generalized Casorati determinant solution generated from

$$\big(\Phi_2^T(\lambda_1), \cdots, \frac{1}{(l_1-1)!}\partial_{\lambda_1}^{l_1-1}\Phi_2^T(\lambda_1); \cdots; \Phi_2^T(\lambda_m), \cdots, \frac{1}{(l_m-1)!}\partial_{\lambda_m}^{l_m-1}\Phi_2^T(\lambda_m)\big)^T$$

is an m-complexiton solution of order $(l_1-1, l_2-1, \cdots, l_m-1)$ to the Toda lattice equation (2.1). Here we clearly see that the order $(l_1-1, l_2-1, \cdots, l_m-1)$ of the complexiton is a sequence of the maximum orders of derivatives with respect to the eigenvalues. The solution generated from the set of eigenfunctions (4.1) is a single complexiton of order $l-1$.

However, it is not easy to compute the derivatives of eigenfunctions with respect to eigenvalues. In what follows, to avoid this difficulty, we would like to consider the system of (2.10) and (2.11) whose coefficient matrix consists of the simplified blocks of the following type:

$$\begin{bmatrix} A_j & & & 0 \\ * & A_j & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & A_j \end{bmatrix}_{l_j\times l_j}, \ A_j = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix}, \tag{4.2}$$

where α_j and $\beta_j > 0$ are real constants, and the symbol $*$ denotes an arbitrary entry. Since the Jordan forms of matrices of this type are of the second type, the resulting solutions are, of course, still complexitons. This form of the coefficient matrix looks more complicated but it will bring us convenience in computing concrete examples of complexitons.

Let us start from a set of eigenfunctions defined by (3.18). Taking derivatives of $\Phi_2 = (\phi_1, \phi_2)^T$ with respect to one of the two constants a and b leads to

$$S\begin{bmatrix} \Phi_2 \\ \frac{1}{1!}\partial_\xi \Phi_2 \\ \vdots \\ \frac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2 \end{bmatrix} = \begin{bmatrix} A & & & 0 \\ \frac{1}{1!}\partial_\xi A & A & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{(l-1)!}\partial_\xi^{l-1} A & \cdots & \frac{1}{1!}\partial_\xi A & A \end{bmatrix}_{l\times l} \begin{bmatrix} \Phi_2 \\ \frac{1}{1!}\partial_\xi \Phi_2 \\ \vdots \\ \frac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2 \end{bmatrix},$$

and

$$[(\frac{1}{k!}\partial_\xi^k \Phi_2)(n)]_t = (\frac{1}{k!}\partial_\xi^k \Phi_2)(n+\delta), \ 0 \le k \le l-1,$$

where ξ denotes a or b, ∂_ξ is the derivative with respect to ξ, and A is defined by

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 2\cosh a \cos b & -2\sinh a \sin b \\ 2\sinh a \sin b & 2\cosh a\cos b \end{bmatrix}. \tag{4.3}$$

This implies that

$$(\Phi_2^T, \frac{1}{1!}\partial_\xi \Phi_2^T, \cdots, \frac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2^T)^T \tag{4.4}$$

is a special solution to the system

$$S\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_{2l-1}\\ \phi_{2l}\end{bmatrix}=\begin{bmatrix}A & & & 0\\ \frac{1}{1!}\partial_\xi A & A & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{(l-1)!}\partial_\xi^{l-1}A & \cdots & \frac{1}{1!}\partial_\xi A & A\end{bmatrix}_{l\times l}\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_{2l-1}\\ \phi_{2l}\end{bmatrix}, \tag{4.5}$$

and

$$\begin{bmatrix}\phi_1(n)\\ \phi_2(n)\\ \vdots\\ \phi_{2l-1}(n)\\ \phi_{2l}(n)\end{bmatrix}_t=\begin{bmatrix}\phi_1(n+\delta)\\ \phi_2(n+\delta)\\ \vdots\\ \phi_{2l-1}(n+\delta)\\ \phi_{2l}(n+\delta)\end{bmatrix}, \tag{4.6}$$

the Jordan form of whose coefficient matrix is of the second type.

Therefore, for $\xi=a$ and $\xi=b$, we obtain two Casorati determinant solutions to the Toda lattice equation (2.1):

$$\begin{cases} a_n=1+\dfrac{d^2}{dt^2}\log \mathrm{Cas}\big(\Phi_2^T(n),\dfrac{1}{1!}\partial_\xi\Phi_2^T(n),\cdots,\dfrac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2^T(n)\big)\,,\\ b_n=\dfrac{d}{dt}\log\dfrac{\mathrm{Cas}\big(\Phi_2^T(n),\dfrac{1}{1!}\partial_\xi\Phi_2^T(n),\cdots,\dfrac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2^T(n)\big)}{\mathrm{Cas}\big(\Phi_2^T(n+1),\dfrac{1}{1!}\partial_\xi\Phi_2^T(n+1),\cdots,\dfrac{1}{(l-1)!}\partial_\xi^{l-1}\Phi_2^T(n+1)\big)}\,,\end{cases} \tag{4.7}$$

which correspond to the simplified blocks of the type in (4.2). Noting that ϕ_1 and ϕ_2 are given explicitly by (3.18), it is direct to compute the derivatives $\partial_a^k\Phi_2$ and $\partial_b^k\Phi_2$, $k\ge 0$, and further the corresponding complexitons. Of course, from pairs of eigenfunctions $(\phi_1(a_j,b_j),\phi_2(a_j,b_j))$, $1\le j\le m$, associated with different complex values $\mu_j=a_j+b_ji$, $1\le j\le m$, two specific m-complexitons of order $(l_1-1,l_2-1,\cdots,l_m-1)$ to the Toda lattice equation (2.1) can be similarly constructed by taking derivatives of the eigenfunctions with the involved pairs of two constants a_j and b_j. Two columns of eigenfunctions required in those two complexitons are

$$\big(\Phi_2^T(a_1,b_1),\cdots,\frac{1}{(l_1-1)!}\partial_{\xi_1}^{l_1-1}\Phi_2^T(a_1,b_1);\cdots;\Phi_2^T(a_m,b_m),\cdots,\frac{1}{(l_m-1)!}\partial_{\xi_m}^{l_m-1}\Phi_2^T(a_m,b_m)\big)^T,$$

where $\xi_j=a_j$ or $\xi_j=b_j$, $1\le j\le m$, and $\Phi_2(a_j,b_j)=(\phi_1(a_j,b_j),\phi_2(a_j,b_j))^T$, $1\le j\le m$, are defined by the formula (3.18) with $a=a_j$ and $b=b_j$. This presents a large class of real complexitons to the Toda lattice equation (2.1).

More generally, we can solve the the system of (4.5) and (4.6) to get a broader set of eigenfunctions required in complexitons. Let us still adopt

$$\lambda=\alpha+\beta i,\ F_k=\phi_{2k-1}+\phi_{2k}i,\ 1\le k\le l,$$

as in Section 3, then the system of (4.5) and (4.6) is equivalent to the following triangular system for all F_k:

$$SF_k=\lambda F_k+\sum_{p=1}^{k-1}\frac{1}{p!}\frac{\partial^p\lambda}{\partial\xi^p}F_{k-p},\ (F_k(n))_t=F_k(n+\delta),\ 1\le k\le l, \tag{4.8}$$

where $F_0=0$. This system contains the system (3.4) as a special case.

Theorem 4.1. *Let $\lambda=\lambda(\xi)$ be a function from $\mathbb{C}$ to $\mathbb{C}-\{2\}$ and $\delta=\pm1$. Suppose that $f(\lambda;c,d)$ is the general solution to (3.5), defined by (3.6), and for $1\le k\le l$, define $f_k=f(\lambda(\xi);c_k,d_k)$ with a pair of arbitrary constants c_k and d_k. Then the general solution to the general triangular system of differential-difference equations (4.8) is given by*

$$F_k=\sum_{p=0}^{k-1}\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial\xi^p}=\sum_{p=0}^{k-1}\frac{1}{p!}\frac{\partial^p f(\lambda(\xi);c_{k-p},d_{k-p})}{\partial\xi^p},\ 1\le k\le l. \tag{4.9}$$

Proof. The proof is similar to the one of Theorem 3.1. Note that the general solution expression (4.9) can be rewritten as

$$F_k=F_{k,h}+F_{k,s},\ F_{k,h}=f_k,\ F_{k,s}=\sum_{p=1}^{k-1}\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial\xi^p},\ 1\le k\le l.$$

By Lemma 3.1, $F_{k,h}=f_k$ is the general solution to the homogeneous counterpart of the k-th subsystem for F_k in (4.8). Use the same argument as in the proof of Theorem 3.1, what we need to prove now is that $F_{k,s}$ is a special solution to the k-th subsystem for F_k in (4.8).

At the current situation, from

$$Sf_k=\lambda(\xi)f_k,\ (f_k(n))_t=f_k(n+\delta),\ 1\le k\le l,$$

we have

$$\begin{cases} S(\dfrac{1}{p!}\dfrac{\partial^p f_k}{\partial\xi^p})=\displaystyle\sum_{q=0}^{p}\frac{1}{q!}\frac{\partial^q\lambda(\xi)}{\partial\xi^q}\frac{1}{(p-q)!}\frac{\partial^{p-q}f_k}{\partial\xi^{p-q}},\ 1\le k\le l,\ p\ge1,\\ \big[(\dfrac{1}{p!}\dfrac{\partial^p f_k}{\partial\xi^p})(n)\big]_t=(\dfrac{1}{p!}\dfrac{\partial^p f_k}{\partial\xi^p})(n+\delta),\ 1\le k\le l,\ p\ge1,\end{cases} \tag{4.10}$$

the former equality of which is equivalent to

$$(S-\lambda(\xi))(\frac{1}{p!}\frac{\partial^p f_k}{\partial \xi^p}) = \sum_{q=1}^{p} \frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}\frac{1}{(p-q)!}\frac{\partial^{p-q} f_k}{\partial \xi^{p-q}},\ 1\le k\le l,\ p\ge 1. \tag{4.11}$$

Therefore, using (4.11), we can compute that

$$\begin{aligned}
(S-\lambda(\xi))F_{k,s} &= \sum_{p=1}^{k-1}(S-\lambda(\xi))\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \xi^p} \\
&= \sum_{p=1}^{k-1}\sum_{q=1}^{p}\frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}\frac{1}{(p-q)!}\frac{\partial^{p-q} f_{k-p}}{\partial \xi^{p-q}} \\
&= \sum_{q=1}^{k-1}\sum_{p=q}^{k-1}\frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}\frac{1}{(p-q)!}\frac{\partial^{p-q} f_{k-p}}{\partial \xi^{p-q}} \\
&= \sum_{q=1}^{k-1}\frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}\sum_{p=q}^{k-1}\frac{1}{(p-q)!}\frac{\partial^{p-q} f_{k-p}}{\partial \xi^{p-q}} \\
&= \sum_{q=1}^{k-1}\frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}\sum_{p=0}^{(k-q)-1}\frac{1}{p!}\frac{\partial^{p} f_{(k-q)-p}}{\partial \xi^{p}} \\
&= \sum_{q=1}^{k-1}\frac{1}{q!}\frac{\partial^q \lambda(\xi)}{\partial \xi^q}F_{k-q},
\end{aligned}$$

and by using the latter equality of (4.10), we have

$$\begin{aligned}
(F_{k,s}(n))_t &= \sum_{p=1}^{k-1}\big[(\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \xi^p})(n)\big]_t \\
&= \sum_{p=1}^{k-1}(\frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial \xi^p})(n+\delta) = F_{k,s}(n+\delta).
\end{aligned}$$

Therefore, the above function $F_{k,s}$ is a special solution to the k-th subsystem of (4.8), indeed. The proof is finished. □

This theorem provides a general set of eigenfunctions required in complexitons, while using (4.5) and (4.6). Now it is just a direct computation to construct a complexiton solution from a τ-function τ_n.

In particular, we can start from the pair of eigenfunctions ϕ_1 and ϕ_2 defined by (3.18) to compute examples of complexitons. First without computing derivatives of

ϕ_1 and ϕ_2, the τ-function of a single complexiton can be expressed as

$$\begin{aligned}\tau_n &= \mathrm{Cas}(\phi_1(n),\phi_2(n)) \\ &= 2cd\,\mathrm{e}^{2t\cosh\delta a\cos\delta b}\sin(2nb+b+2t\cosh\delta a\sin\delta b)\sinh a \\ &\quad +c^2\mathrm{e}^{2na+a+2t\mathrm{e}^{\delta a}\cos\delta b}\sin b-d^2\mathrm{e}^{-2na-a+2t\mathrm{e}^{-\delta a}\cos\delta b}\sin b\,, \end{aligned} \tag{4.12}$$

where $\delta=\pm1$ and a,b,c,d are arbitrary real constants, but $b\neq0$ and $c^2+d^2\neq0$ in order that $\tau_n\neq0$. If we fix $c=\pm d$, and the τ-function boils down to

$$\begin{aligned}\tau_n &= 2c^2\,\mathrm{e}^{2t\cosh\delta a\cos\delta b}\sinh(2na+a+2t\sinh\delta a\cos\delta b)\sin b \\ &\quad \pm2c^2\,\mathrm{e}^{2t\cosh\delta a\cos\delta b}\sin(2nb+b+2t\cosh\delta a\sin\delta b)\sinh a\,. \end{aligned} \tag{4.13}$$

Second, through computing the first-order derivatives of ϕ_1 and ϕ_2, the τ-function of a single complexiton of order 1 reads as

$$\tau_n=\mathrm{Cas}\big(\phi_1(n),\phi_2(n),\frac{\partial\phi_1(n)}{\partial\xi},\frac{\partial\phi_2(n)}{\partial\xi}\big), \tag{4.14}$$

where $\xi=a$ or $\xi=b$. More generally, upon choosing arbitrary real constants a_i,b_i, $i=1,2$ and c_i,d_i, $1\le i\le3$, which satisfy $b_i\neq0$, $i=1,2$ and $c_i^2+d_i^2\neq0$, $1\le i\le3$, we can have a τ-function of a single complexiton of order 1:

$$\begin{aligned}\tau_n=\mathrm{Cas}\big(&(\phi_1(a_1,b_1;c_1,d_1))(n),(\phi_2(a_1,b_1;c_1,d_1))(n),\\ &(\phi_1(a_1,b_1;c_2,d_2))(n)+\frac{\partial(\phi_1(a_1,b_1;c_1,d_1))(n)}{\partial\xi_1},\\ &(\phi_2(a_1,b_1;c_2,d_2))(n)+\frac{\partial(\phi_2(a_1,b_1;c_1,d_1))(n)}{\partial\xi_1}\big),\end{aligned} \tag{4.15}$$

and a τ-function of a 2-complexiton of order (1,1):

$$\begin{aligned}\tau_n=\mathrm{Cas}\big(&(\phi_1(a_1,b_1;c_1,d_1))(n),(\phi_2(a_1,b_1;c_1,d_1))(n),\\ &\frac{\partial(\phi_1(a_1,b_1;c_1,d_1))(n)}{\partial\xi_1},\frac{\partial(\phi_2(a_1,b_1;c_1,d_1))(n)}{\partial\xi_1},\\ &(\phi_1(a_2,b_2;c_2,d_2))(n),(\phi_2(a_2,b_2;c_2,d_2))(n),\\ &(\phi_1(a_2,b_2;c_3,d_3)(n)+\frac{\partial(\phi_1(a_2,b_2;c_2,d_2))(n)}{\partial\xi_2},\\ &(\phi_2(a_2,b_2;c_3,d_3)(n)+\frac{\partial(\phi_2(a_2,b_2;c_2,d_2))(n)}{\partial\xi_2}\big),\end{aligned} \tag{4.16}$$

where $\xi_i=a_i$ or $\xi_i=b_i$, $i=1,2$.

5 Concluding remarks

A set of coupled conditions consisting of differential-difference equations has been proposed for Casorati determinants to solve the Toda lattice equation. A systematic analysis has been made for solving the resulting system of differential-difference equations whose coefficient matrix consists of Jordan blocks of the second type, together with the solution formula for the key subsystem associated with one Jordan block. The resulting set of eigenfunctions leads to complexitons through the Casoratian formulation. Moreover, a feasible way has been presented to construct sets of eigenfunctions required for forming complexitons, which allows us to directly compute examples of real complexitons.

We remark that the resulting complexitons of order zero (i.e., not involving derivatives of eigenfunctions) can be constructed from complexification of wave numbers of 2-solitons (see [15] for the KdV case). However, the resulting complexitons of order larger than zero (i.e., involving derivatives of eigenfunctions) can not be generated from complexification of solitons. Such solutions are generated on the basis of our coupled conditions established in Theorem 2.1. On the other hand, based on Theorem 2.1, our generalized Casorati determinant solutions give solitons and negatons if $\lambda > 2$, positons if $\lambda < 2$ and rational solutuons to the Toda lattice equation if $\lambda = 2$ [7, 14]. Viewing $(S-2)\phi$ as a discrete version of $\partial_x^2\phi$, we can easily see that this is consistent with the phenomenon in the KdV case [11].

Our results also indicate that integrable equations can have three different kinds of explicit exact transcendental function solutions: negatons, positons and complexitons. Solitons are usually a specific class of negatons. Roughly speaking, negatons and positons are solutions which involve exponential functions and trigonometric functions of space variables, respectively, and they are all associated with real eigenvalues of the associated spectral problems. But complexitons are different solutions which involve both exponential and trigonometric functions of space variables, and they are associated with complex eigenvalues of the associated spectral problems. Interaction solutions among negatons, positons, rational solutions and complexitons are a class of much more general and complicated solutions to soliton equations, in the category of elementary function solutions. There is also a large class of θ-function solutions to soliton equations. It is an interesting question for us what inverse scattering data there exist for complexitons of the Toda lattice equation.

It is also natural to ask whether our idea of constructing complexitons can be

successfully applied to other integrable lattice equations such as the Ablowitz-Ladik (AL) equation [16, 17] and general Toda lattice equations [18]. Particularly interesting to us is to make an extension to full discrete integrable equations such as the discrete-time KdV equation [19] and the discrete-time Toda lattice equation [20]. On the other hand, it has been pointed out that multi-positon solutions of the KdV equation may be related to giant ocean waves such as "freak wave" (rogue wave), breaking up ships [21]. It is our hope that complexitons can provide certain mathematical background for related nonlinear phenomena in the field of mathematical physics.

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